Particle Cosmology and Baryonic Astrophysics
Part I

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Contents

The first part of the lecture will follow Dodelson, Modern Cosmology [1] very closely. Throughout the notes, we will use natural units:

$$\hbar = c = k_B = 1 \,.$$

1 The Basic Ingredients of the Universe

See also Dodelson, Modern Cosmology [1], chapter 2 and Kolb/Turner, The Early Universe [2], chapter 1-3.

Hubble discovered in 1929[3] that distant galaxies are moving away from us. His observation is shown in Fig. 1. From this diagram, we can extract the slope, called *Hubble rate* H_0, today,

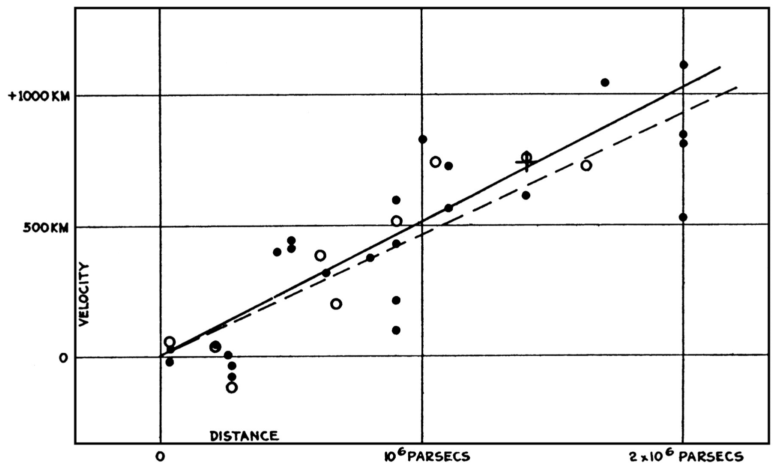

Figure 1: Hubble diagram: velocity — distance relation among extra-galactic nebulae. The velocity is in $\mathrm{km\,sec}^{-1}$ and the distance in Mpc.

$$H_0 = 100\, h \mathrm{km\,sec}^{-1}\,\mathrm{Mpc}^{-1} \ . \tag{1.1}$$

The Planck satellite mission measured $H_0 = (67.1 \pm 1.2)\ \mathrm{km\,sec}^{-1}\,\mathrm{Mpc}^{-1}$ [4].

1.1 Metric

In order to understand the Hubble diagram, we have to learn how to measure distances and length scales in the Universe. Before looking at distances in space-time, let us first consider distances in Euclidean space. In Euclidean space, the distance between two points is given by the distance in x and y direction between the two points in Cartesian coordinates

$$ds^2 = dx^2 + dy^2 \tag{1.2}$$

where we used Cartesian coordinates to write the distance in the last term. However the result should not depend on the chosen coordinate system. Thus choosing polar coordinates ($r = \sqrt{x^2 + y^2}$, θ) with

$$x = r \sin\theta \qquad\qquad y = r\cos\theta \ , \tag{1.3}$$

we find for a distance between two points

$$ds^2 = dr^2 + r^2 d\theta^2 \ . \tag{1.4}$$

In general we can write

$$ds^2 = \sum_{ij} g_{ij} dx^i dx^j \ , \tag{1.5}$$

where g is a symmetric matrix, which is called *metric*. The metric defines a scalar product on the vector space and consequently a norm, which can be used to define distances. In four space-time dimensions, we conventionally write

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{1.6}$$

The $\sum$ sign is often dropped and it is convention to sum over the same index, if it appears as lower and upper index. ds^2 is sometimes called *proper time*. The metric g has 10 degrees of freedom.

One special case is special relativity with the metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \,. \tag{1.7}$$

The *signature* of the metric is the number of eigenvalues ± 1 of the metric. In case of special and general relativity it is $(3, 1)$ or $(1, 3)$ depending on the convention whether the time component has eigenvalue ± 1.

At large scales, the Universe appears homogeneous and isotropic. In addition, we find that the is flat. The space-time is described by the Friedmann-Robertson-Walker (FRW) metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a(t)^2 & & \\ & & a(t)^2 & \\ & & & a(t)^2 \end{pmatrix} \,. \tag{1.8}$$

if the Universe is flat, or more generally by

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \tag{1.9}$$

for a general homogeneous and isotropic metric, where $k = 0$ for a flat Universe, $k = 1$ for a closed Universe, $k = -1$ for an open Universe. We will be mainly looking at a flat Universe with $k = 0$. See the assignment for the derivation of this metric. The parameter a is called scale factor and describes how the Universe expands. See Fig. 2.

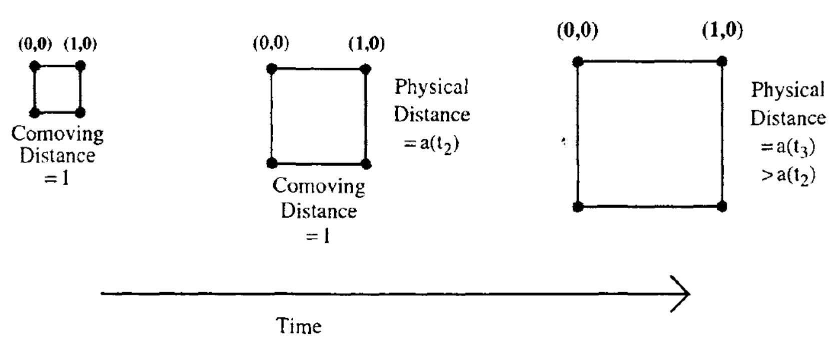

Figure 2: Expansion in an FRW Universe. Copied from [1]

1.2 Geodesics

How does a particle move without any external forces? Newton's law tells us

$$\frac{d^2x^i}{dt^2} = 0\ . \tag{1.10}$$

How can we generalise this to a general coordinate system? For example for a system in polar

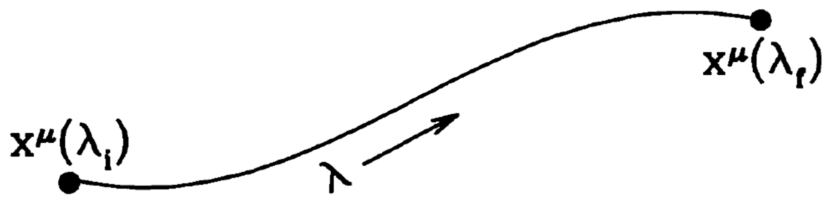

Figure 3: Curve in space-time. Copied from [1]

coordinates, $x' = (r, \theta)$, the equations of motion look different. Starting from a Cartesian coordinate system, we find

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\ . \tag{1.11}$$

with the *transformation matrix* $\partial x^i/\partial x'^j$. In case of polar coordinates

$$x^1 = x'^1 \cos x'^2 \qquad\qquad x^2 = x'^1 \sin x'^2 \tag{1.12}$$

the transformation matrix is

$$\frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \cos x'^2 & -x'^1 \sin x'^2 \\ \sin x'^2 & x'^1 \cos x'^2 \end{pmatrix}\ . \tag{1.13}$$

Applying the second derivative and doing the algebra we find

$$0 = \frac{d^2x^i}{dt^2} = \frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = \frac{\partial x^i}{\partial x'^j}\frac{d^2x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} \tag{1.14}$$

multiplying with the inverse of the transformation matrix we obtain

$$\frac{d^2x'^l}{dt^2} + \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^l_{\ i} \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0\ . \tag{1.15}$$

Solutions to the this equation are called *geodesics* and the equation itself is commonly denoted by *geodesic equation*. There are two small changes in general relativity, the index runs from 0 to 3 and we can not use time t to parameterize the path, but we have to use different monotonically increasing parameter along the geodesic. With these modifications we can rewrite the geodesic equation as

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{1.16}$$

with the *Christoffel symbol*

$$\Gamma^\mu_{\alpha\beta} = \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^\mu_{\ \kappa} \frac{\partial^2 x^\kappa}{\partial x'^\alpha \partial x'^\beta} \tag{1.17}$$

A more convenient form of the Christoffel symbol is in terms of the metric tensor

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\ . \tag{1.18}$$

1.3 Point Particle in FRW Universe

Let us study a massless point particle in an FRW Universe.

The energy-momentum four-vector $P = (E, \vec{P})$ of point particle can be used to define the parameter λ by

$$P^\alpha = \frac{dx^\alpha}{d\lambda} \ . \tag{1.19}$$

Looking at the 0-component we can reexpress the derivative

$$\frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt} \ . \tag{1.20}$$

The Christoffel symbol for a FRW metric has only a few non-vanishing components:

$$\Gamma^0_{ij} = \delta_{ij}\dot{a}a \qquad\qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\frac{\dot{a}}{a} \ , \tag{1.21}$$

which can be easily derived using Eqs. (1.18) and (1.9) for $k = 0$. Hence the zeroth component of the geodesic equation is given by

$$\frac{d^2x^0}{d\lambda^2} = \frac{d}{d\lambda}\frac{dx^0}{d\lambda} = E\frac{dE}{dt} = -\delta_{ij}\dot{a}aP^iP^j \ . \tag{1.22}$$

Using the on-shell condition of the particle

$$0 = g_{\mu\nu}P^\mu P^\nu = -E^2 + \delta_{ij}a^2P^iP^j \tag{1.23}$$

we find that the geodesic equation implies

$$\frac{dE}{dt} + \frac{\dot{a}}{a}E = 0 \quad \Rightarrow \quad E \propto \frac{1}{a} \ . \tag{1.24}$$

1.4 Redshift

Galaxies further away are moving away from us. Hence the light we observe is red-shifted compared to the emitted light. We can define

$$1 + z \equiv \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{1}{a} \ . \tag{1.25}$$

For small velocities $v \ll c$, the standard redshift formula can be used and we obtain $z \simeq v/c$. It is thus a direct measure of the velocity of the galaxies.

1.5 Distances

There are two ways to measure distance, the comoving distance, χ, which remains fixed during expansion, and the physical distance, $d = a\chi$, which takes the expansion into account. As we are in an expanding space-time, we might wonder what is the more interesting physical distance: the distance at the time when the light was emitted or the distance when it was received. The well-defined measure of distance is a comoving distance. On a comoving grid, the distance simply amounts to $\left(dx^2 + dy^2 + dz^2\right)^{1/2}$. See Fig. 2 for an illustration.

1.5.1 Comoving distances

In any given time dt, light travels a comoving distance $dx = dt/a$.

The total comoving distance is given by the distance light could have travelled in a given time t, i.e.

$$\eta(t) = \int_0^t \frac{dt'}{a(t')} \; . \tag{1.26}$$

As nothing travels faster than light, $\eta(t)$ defines the *particle horizon*. We are not able to see anything in the past, which is beyond the particle horizon. It is monotonically increasing and can also be used as a measure of time, the so-called *conformal time*. The proper distance of the particle horizon is given by

$$d_{max}(t) = a(t) \int_0^t \frac{dt'}{a(t')} \; . \tag{1.27}$$

Similarly, there might be a horizon for future events, if the universe recollapses at time T. Then the largest distance from which an observer might be able to receive signals travelling at the speed of light at any time later than t, is given by

$$\int_t^T \frac{dt'}{a(t')} \tag{1.28}$$

in comoving coordinates, which is denoted *event horizon*. The proper distance for an infinite distant future is given by

$$d_{MAX}(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} \; . \tag{1.29}$$

Generally we can define the comoving distance of an object at time $t(a)$ to today

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} \tag{1.30}$$

with the *Hubble rate*

$$H \equiv \frac{\dot{a}}{a} \; . \tag{1.31}$$

1.5.2 Angular diameter distance

A common method to determine the distance of an object of known size l is to measure the angle θ it takes on the sky. Then the *angular diameter distance* of the object is given by

$$d_A \equiv \frac{l}{\theta} \tag{1.32}$$

In an expanding Universe, we have to take the expansion into account. The comoving size of the object is given by l/a and its angle in the sky $\theta = (l/a)/\chi(a)$. Thus we can reexpress the angular diameter distance in a flat FRW space-time as

$$d_A^{flat} = a\chi(a) = \frac{\chi}{1+z} \; . \tag{1.33}$$

Similar expressions can be derived for an open ($k = -1$) or closed ($k = 1$) space-time.

1.5.3 Luminosity Distance

Another common way to infer the distance to an object of known luminosity is to compare the known to the measured luminosity L. Neglecting the expanding space-time, the observed flux F at a distance d is given by

$$F = \frac{L}{4\pi d^2} \,. \tag{1.34}$$

Generalising the expression to comoving coordinates, we find

$$F = \frac{L(\chi)}{4\pi\chi^2(a)} \,. \tag{1.35}$$

At early times, the photons travel further on the comoving grid, then at later times. Hence we find due to the expansion the number of photons crossing a shell at distance χ is less by a factor a. If the photons are all emitted with the same energy, then the luminosity which we observe today is smaller by a factor a^2, because the photons are red-shifted in addition.

$$F = \frac{La^2}{4\pi\chi^2(a)} \,. \tag{1.36}$$

A comparison with Eq. (1.34) shows that the apparent distance is defined by the *luminosity distance*

$$d_L \equiv \frac{\chi}{a} \,. \tag{1.37}$$

1.6 Covariant Derivative

Before continuing let us introduce one more concept. The four-vector along the path of a massive particle $\frac{dx^\alpha}{d\lambda}$ is a tangent vector at any given point along the path. We already saw how a tangent vector transforms under a general coordinate transformation in Eq. (1.11) using the chain rule

$$\frac{dx^\mu}{d\lambda} = \left(\frac{\partial x}{\partial x'}\right)^\mu_{\ \nu} \frac{dx'^\nu}{d\lambda} \,. \tag{1.38}$$

How does $\partial_\mu \frac{dx^\mu}{d\lambda}$ transform under a general coordinate transformation? It is not invariant, because ∂_μ also acts on $\left(\frac{\partial x}{\partial x'}\right)$. In order to obtain an invariant quantity, we have to define the covariant derivative of an arbitrary vector V^ν in the tangent space

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \,. \tag{1.39}$$

The expression in Eq. (1.18) is a result of requiring that the covariant derivative (connection) is metric compatible $\nabla_\rho g_{\mu\nu} = 0$.

1.7 Einstein Equations

The metric introduced in the previous sections describes gravity and the interaction of gravity with matter is described by Einstein equation[1]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{1.41}$$

[1]A possible cosmological constant Λ is absorbed in the energy-momentum tensor, i.e.

$$T^\Lambda_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \,. \tag{1.40}$$

with Newton's constant G, the Einstein tensor $G_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $\mathcal{R}$, and the energy-momentum tensor $T_{\mu\nu}$. The Ricci tensor and Ricci scalar describe the curvature of space-time. The Ricci scalar is simply defined by the contraction of the Ricci tensor with the metric

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} \tag{1.42}$$

and the Ricci tensor can be obtained from the Christoffel symbols[2]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \, . \tag{1.43}$$

In particular looking at the 00 component of the Ricci tensor in an FRW metric we find

$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} && (1.44) \\ &= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} && (1.45) \\ &= -\frac{\partial}{\partial t}\delta_{ii}\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \delta_{ij}\delta_{ij} = -3\frac{\ddot{a}}{a} \, . && (1.46) \end{aligned}$$

Similarly for the spatial components we find

$$R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \tag{1.47}$$

and the Ricci scalar can be evaluated to

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{1}{a^2}R_{ii} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) \, . \tag{1.48}$$

Plugging everything into Einstein equations we obtain two independent equations

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}\mathcal{R} = 3\left(\frac{\dot{a}}{a}\right)^2 &= 8\pi G T_{00} && (1.49) \\ g^{\mu\nu}G_{\mu\nu} = -\mathcal{R} &= 8\pi G T^\mu_\mu \end{aligned}$$

1.8 Perfect Fluid

Before interpreting the Einstein equations, we have to have a closer look at the energy-momentum tensor $T^{\mu\nu}$. Our basic assumption is that we can describe the content of the Universe by different perfect fluids as a leading approximation, i.e. the fluid can be described by macroscopic quantities, its energy density and pressure, while there is no stress or viscosity in agreement that with the metric being homogeneous and isotropic.

The energy momentum tensor describes the flux of four-momentum p^μ in the direction x^ν. The energy-momentum tensor of a perfect fluid in its rest-frame in Minkowski space is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \tag{1.50}$$

[2]The curvature is defined similar to the field strength tensor in quantum field theory from the commutator of the covariant derivatives $[\nabla_\mu, \nabla_\nu]$. Please refer to a general relativity book.

Due to isotropy it is diagonal and its spatial components have to be equal. The 00-element is just the energy density ρ, i.e. the flux of energy in time direction, while the spatial elements ii are given by the flux of momentum density p_i in the direction x_i, i.e. the pressure $\mathcal{P}_i = \frac{dp_i}{dt} dx_i$ in direction x_i. In order to write it in a covariant form, we first introduce the four-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \tag{1.51}$$

with the *proper time*

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.52}$$

For a particle at rest we find $U^\mu = (1, 0, 0, 0)$. Hence we can write the energy-momentum tensor as

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P} \eta^{\mu\nu} \tag{1.53}$$

and its generalisation to general relativity is straightforward

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P} g^{\mu\nu} \ . \tag{1.54}$$

Thus we find in the rest-frame of the fluid

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}\mathcal{P} & & \\ & & a^{-2}\mathcal{P} & \\ & & & a^{-2}\mathcal{P} \end{pmatrix} \qquad \text{or} \qquad T^\mu_\nu = \begin{pmatrix} -\rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \ . \tag{1.55}$$

For example dust can be described by a perfect fluid with zero pressure, since it is not compressible.

1.9 Friedmann Equations

Using our knowledge about the energy-momentum tensor of a perfect fluid, we see that

$$T_{00} = \rho \qquad\qquad T^\mu_\mu = -\rho + 3\mathcal{P} \tag{1.56}$$

and we can rewrite Eqs. (1.49) to obtain the Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{1.57}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\mathcal{P}) \ . \tag{1.58}$$

1.10 Continuity Equation

How does the energy-momentum tensor of the perfect fluid evolve with time? In the absence of external forces and gravity, we find that the energy density is constant $\partial\rho/\partial t = 0$ and the Euler equation that the pressure does not depend on the direction $\partial\mathcal{P}/\partial x^i$. In covariant formulation, this amounts to

$$\partial_\mu T^\mu_\nu = 0 \tag{1.59}$$

which some might have seen in the quantum field theory course. The generalisation to general relativity is straightforward by understanding that we have to replace the partial derivative with a covariant

	w	$\rho(a)$	$a(t)$	$H(t)$
matter	0	a^{-3}	$t^{2/3}$	$\frac{2}{3t}$
radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	$\frac{1}{2t}$
cosm.const.	-1	ρ_0	$e^{\sqrt{\Lambda/3}t}$	$\sqrt{\Lambda/3}$

Table 1: Evolution of different fluids

derivative to ensure that the continuity equation correctly transforms under a change of coordinates, i.e.

$$0 = \nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha \; . \tag{1.60}$$

For $\nu = 0$ we obtain

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\alpha\mu} T^\alpha_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha \tag{1.61}$$

$$= -\frac{\partial\rho}{\partial t} - \Gamma^i_{0i}\rho - \Gamma^i_{0i} T^i_i \tag{1.62}$$

and thus

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}\left[\rho + \mathcal{P}\right] = 0 \; . \tag{1.63}$$

Introducing the *equation of state*

$$\mathcal{P} = w\rho \tag{1.64}$$

we can rewrite the continuity equation for $\nu = 0$

$$0 = \frac{\partial\rho}{\partial t} + 3(1+w)\frac{\dot{a}}{a}\rho = a^{-3(1+w)}\frac{\partial\left(\rho a^{3(1+w)}\right)}{\partial t} \tag{1.65}$$

for constant equation of state parameter w and conclude $\rho \propto a^{-3(1+w)}$. We can insert this result into the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \; . \tag{1.66}$$

The solution gives us the time dependence of the scale factor for $w \neq -1$

$$a(t) \propto t^{\frac{2}{3(1+w)}} \qquad\qquad H = \frac{2}{3(1+w)t} \tag{1.67}$$

Matter has $w = 0$ and thus $\rho \propto a^{-3}$, $a(t) \propto t^{2/3}$ and $H = 2/3t$, radiation on the other hand has $w = 1/3$ and thus $\rho \propto a^{-4}$, $a(t) \propto t^{1/2}$ and $H = 1/2t$. For a cosmological constant with $w = -1$, the Friedmann equation is particularly simple

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \tag{1.68}$$

and thus $a \propto e^{\sqrt{\Lambda/3}t}$. See Tab. 1.

2 Thermodynamics of the Early Universe

2.1 Equilibrium Thermodynamics

Usually interaction in the Early Universe happen quickly enough to keep the particles in local thermal equilibrium and different fluids share the same temperature. It is convenient to use distribution functions $f(\vec{x},\vec{p})$, i.e. the occupation number of a small cell $d^3x d^3p/(2\pi\hbar)^3$ at position $(\vec{x},\vec{p})$ to describe the fluid. The number density n_i of species i with g_i internal degrees of freedom is given by

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(\vec{x},\vec{p}) \; . \tag{2.1}$$

Bosons and fermions follow the usual Bose-Einstein and Fermi-Dirac distributions in equilibrium at a temperature T respectively

$$f(\vec{x},\vec{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1} \tag{2.2}$$

with $+$ for the Fermi-Dirac and $-$ for the Bose-Einstein distribution. Similarly we can define the energy density and the pressure (See prob. 15 in chapter 2 of [1])

$$\rho = g_i \int \frac{d^3p}{(2\pi)^3} f(\vec{x},\vec{p}) E(p) \xrightarrow{T\gg m} \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{bosons} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{fermions} \end{cases} \tag{2.3}$$

$$\mathcal{P} = g_i \int \frac{d^3p}{(2\pi)^3} f(\vec{x},\vec{p}) \frac{p^2}{3E(p)} \xrightarrow{T\gg m} \frac{1}{3}\rho \; . \tag{2.4}$$

See exercise 15 in chapter 2 of [1] to understand the form of the expressions for the energy density and the pressure. The solutions to the exercise are provided in the appendix of [1]. For negligible chemical potentials

$$d(\rho(T)V) = T d(s(T)V) - \mathcal{P}(T) dV \tag{2.5}$$

allows us to write the entropy density $s(T)$ as

$$s(T) = \frac{\rho(T) + \mathcal{P}(T)}{T} \tag{2.6}$$

by equating the coefficient in front of dV. Similarly it is straightforward to show

$$s(T) = \frac{\partial \mathcal{P}}{\partial T} \tag{2.7}$$

using either one of the Maxwell relations or considering the coefficient in front of the differential $V dT$ in Eq. (2.5). The condition of thermal equilibrium tells us that the entropy in a comoving volume is fixed, i.e.

$$s(T) a^3 = \text{constant} \; . \tag{2.8}$$

See Dodelson pg. 39/40 for a derivation using the continuity equation.

In the radiation dominated era, it is convenient to define the effective relativistic degrees of freedom $g_*^{\rho,s}(T)$ as follows

$$\rho = \frac{\pi^2}{30} g_*^{\rho}(T) T^4 \qquad\qquad s = \frac{2\pi^2}{45} g_*^{s}(T) T^3 \; . \tag{2.9}$$

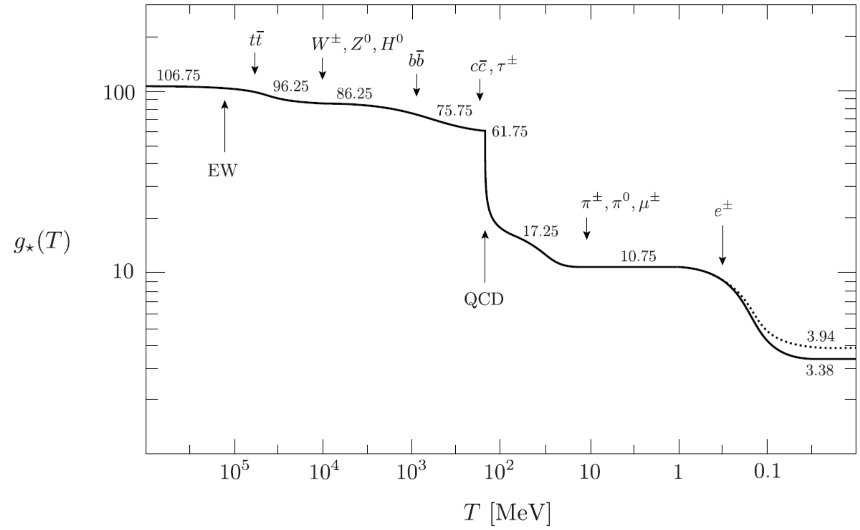

Figure 4: Evolution of effective relativistic degrees of freedom. Solid line is for $g_*^\rho(T)$ and the dotted line for $g_*^s(T)$. Taken from Cosmology lecture notes of Daniel Baumann.

For most of the time, $g_*^\rho(T) = g_*^s(T)$, as it is shown in Fig 4.

See the accompanying slides for a discussion of the different components making up the energy budget in the Universe: dark energy, photons, neutrinos, baryons and dark matter. Most of the discussion is straightforward. For completeness however I would like to repeat the discussion about the temperature of neutrinos today. Neutrinos are almost massless fermions. They decouple from the cosmic plasma around 1 MeV and thus shortly before electrons and positrons become non-relativistic and reheat the cosmic plasma. Thus neutrinos are effectively colder than the cosmic plasma, since they are not reheated by electron-positron pair annihilation. Using entropy conservation, we find for the entropy before neutrino decoupling at scale factor a_1

$$s(a_1) = \frac{2\pi^2}{45} T_1^3 \left[2 + \frac{7}{8}\left(2 + 2 + 3 \cdot 2\right) \right] = \frac{43\pi^2}{90} T_1^3 \,, \tag{2.10}$$

because there are in total 2 degrees of freedom from the two polarisations of photons, 2 spin degrees of freedom for both electrons and positrons and 3 generations of neutrinos with spin 2. After electrons and positrons become non-relativistic, they transfer their entropy to the cosmic plasma and effectively reheat the cosmic plasma. Hence the entropy at a late-enough redshift a_2 is given by

$$s(a_2) = \frac{2\pi^2}{45} \left[2T_\gamma^3 + \frac{7}{8} 6 T_\nu^3 \right] \,, \tag{2.11}$$

where photons have a temperature T_γ and neutrinos have temperature T_ν. Entropy conservation $s(a_1)a_1^3 = s(a_2)a_2^3$ results in

$$\frac{43}{2} \left(a_1 T_1\right)^3 = 4 \left[\left(\frac{T_\gamma}{T_\nu} \right)^3 + \frac{21}{8} \right] \left(T_\nu(a_2) a_2\right)^3 \,. \tag{2.12}$$

Finally we have to relate the temperature T_1 to the temperature at a later time. After neutrinos are decoupled, they still preserve the shape of the Fermi-Dirac distribution and the temperature is

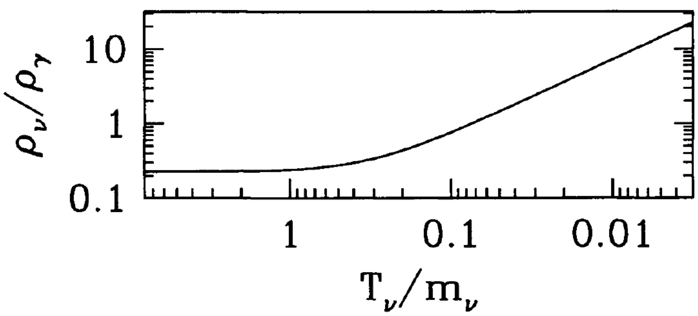

Figure 5: Neutrino energy compared to photon energy vs temperature of neutrinos. Taken from Dodelson[1]

inversely proportional to the scale factor. This can be directly seen from observing that the energy of a massless particle scales like a^{-1} as shown in Eq. (1.24). Thus the temperature of neutrinos T_ν satisfies $a_2 T_\nu = a_1 T_1$. Solving Eq. (2.12) for the temperature of neutrinos T_ν, we obtain

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \tag{2.13}$$

and conclude that the temperature of neutrino background today is lower compared to the cosmic microwave background. We find for the temperature of neutrinos today

$$T_\nu^0 = T_\gamma^0 \left(\frac{4}{11}\right)^{1/3} = 2.73 \left(\frac{4}{11}\right)^{1/3} K = 1.95K = 1.68 \times 10^{-4}\text{eV} . \tag{2.14}$$

It has been undeniably shown that neutrinos are massive. The temperature of neutrinos today T_ν^0 is smaller than the square root of the solar mass squared difference, $\sqrt{\Delta m_\odot^2} = 8.66 \times 10^{-3}$eV, and thus at least two neutrinos are non-relativistic today and their mass can not be neglected. The energy density of one neutrino is given by

$$\rho_{1\nu} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_\nu^2}}{e^{p/T_\nu} + 1} \tag{2.15}$$

and shown in Fig. 5. Thus the total energy density in neutrinos is dominated by their mass $\rho_\nu = m_\nu n_\nu$ and we find using $n_\nu = 3n_\gamma/11$

$$\Omega_\nu h^2 = \frac{m_\nu}{94\text{eV}} . \tag{2.16}$$

2.2 Boltzmann Equation for Number Density

The Boltzmann equation describes the time evolution of the phase space density. We will first concentrate on the integrated form of the Boltzmann equation and study the time evolution of the number densities. For annihilation $1 + 2 \leftrightarrow 3 + 4$, we find for the phase space density of particle 1, n_1

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = \int \prod_{i=1}^{4} d\Pi_i \, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \, |\mathcal{M}|^2 \times \left\{ f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \right\} \tag{2.17}$$

with the phase space integrals

$$\int d\Pi_i = g_i \int \frac{d^3p_i}{(2\pi)^3 2E_i(p_i)} \tag{2.18}$$

for particle i with g_i internal degrees of freedom. The phase space integral is Lorentz invariant

$$\int \frac{d^3p}{2E(p)}\delta(E-\sqrt{p^2+m^2}) = \int d^3p \int_0^\infty dE\delta(E^2-p^2-m^2) \tag{2.19}$$

because we implicitly impose that the particle are on-shell, i.e. satisfy the energy-momentum dispersion relation

$$E^2 = p^2 + m^2 \tag{2.20}$$

Let us understand the different factors in the Boltzmann equation. In the absence of any interactions, the right-hand side of the equation, the Boltzmann equation tells us that the number of particles in a comoving volume does not change. However the number of particles in a physical volume scales like a^{-3} due to the expansion. Interactions between the different particles are described by the integral on the right-hand side. The integrals are over the whole phase space $\int d\Pi_i$ of the different particles involved in the interaction. Energy-momentum conservation is imposed by the four-dimensional delta function. The factor $|\mathcal{M}|^2$ is the square of the amplitude (matrix element), which governs the strength of the interaction. For example in the case of Compton scattering it is proportional to the fine-structure constant α^2. It is averaged over *initial and final states*. The last factor on the right-hand side consists of two terms and takes into account the occupation numbers (distribution functions) of the different states. The first term is proportional to $f_3 f_4(1\pm f_1)(1\pm f_2)$ and describes the production of a particle 1 in the process $3+4\to 1+2$, i.e. it is proportional to the initial abundances and the factors $(1\pm f_i)$ take into account the possible Pauli-blocking for fermions with a minus sign or Bose enhancement for bosons with a plus sign. The second term describes the destruction of particle 1 in the process $1+2\to 3+4$. The first term is sometimes called *source term* and the second *loss term*. Note that we assumed that the process is reversible.

Usually scattering between the different particles enforces kinetic equilibrium, i.e. the different particle species follow the Bose-Einstein or Fermi-Dirac statistics, however they are not necessarily in chemical equilibrium and μ would be the chemical potential, which would have to balance against the other chemical potentials, e.g. for $e^+ + e^- \leftrightarrow \gamma\gamma$, we would find $\mu_{e^+}+\mu_{e^-} = 2\mu_\gamma$.

For systems at temperature $T \ll E-\mu$ we can neglect the terms ± 1 in the denominators of the Fermi-Dirac and Bose-Einstein distributions and work with the Maxwell-Boltzmann distribution

$$f(E) = e^{-(E-\mu)/T} = e^{\mu/T}e^{-E/T}\ . \tag{2.21}$$

Similarly we can neglect the Pauli-blocking/Bose enhancement factors and can approximate

$$\begin{aligned}&\{f_3 f_4 (1\pm f_1)(1\pm f_2) - f_1 f_2 (1\pm f_3)(1\pm f_4)\}\\ &\qquad \to f_3^{MB} f_4^{MB} - f_1^{MB} f_2^{MB} = e^{-(E_1+E_2)/T}\left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\right)\end{aligned} \tag{2.22}$$

using energy-momentum conservation. The number density

$$n_i = n_i^{(0)} e^{\mu_i/T} \tag{2.23}$$

of species i can be expressed as a function of μ_i and the *equilibrium number density*

$$n_i^{(0)} = g_i \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} = \begin{cases} g_i\left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} & m_i \gg T \\ \zeta(3) g_i \frac{T^3}{\pi^2} & m_i \ll T \end{cases}\ . \tag{2.24}$$

Using this we can rewrite Eq. (2.22)

$$e^{-(E_1+E_2)/T}\left\{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right\} \tag{2.25}$$

and consequently the Boltzmann equation

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v\rangle \left\{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right\} \tag{2.26}$$

where we defined the thermally averaged cross section

$$\langle \sigma v\rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int d\Pi_i e^{-(E_1+E_2)/T} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4) \left|\mathcal{M}\right|^2 \ . \tag{2.27}$$

Before moving on, a few comments are in order. Note that we could equally well use $E_3 + E_4$ and the equilibrium number densities of the particles 3 and 4. It is straightforward to generalise the expression to other processes, like decays ($1 \to 2$) processes or scattering with more than 2 particles in the final state.

The Boltzmann equation can be applied to many processes in the early Universe. We will discuss the freeze-out of a massive particle, which is relevant for dark matter production, in detail, and outline how it can be applied to Big-Bang Nucleosynthesis (BBN) and recombination.

If the interaction rate $\langle \sigma v\rangle n_2^{(0)}$ is large compared to the Hubble rate, the Boltzmann equation (2.26) can only be satisfied if the number densities satisfy

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{2.28}$$

and consequently the chemical potential are related by

$$\mu_3 + \mu_4 = \mu_1 + \mu_2 \ . \tag{2.29}$$

This case is commonly denoted *chemical equilibrium* in the context of the production of heavy relics, *nuclear statistical equilibrium* in the context of big bang nucleosynthesis, and *Saha equation* in the context of recombination.

2.3 Freeze-Out

The prime example for freeze-out is dark matter production via freeze-out. We consider a massive Dirac particle X with mass m_X, which is initially in thermal equilibrium with the cosmic plasma, but later freezes-out, i.e. decouples from the thermal SM plasma. Let us consider processes of the type $X\bar{X} \leftrightarrow l\bar{l}$, where the pair of particles $X\bar{X}$ annihilate into a pair of light particles $l\bar{l}$ and vice versa. We assume that the light particle is in chemical as well as kinetic equilibrium with the cosmic plasma, i.e. $n_l = n_l^{(0)}$. Thus we find for the Boltzmann equation of $n_X = n_{\bar{X}}$ (2.26)

$$a^{-3}\frac{d\left(n_X a^3\right)}{dt} = \langle \sigma v\rangle \left\{\left(n_X^{(0)}\right)^2 - n_X^2\right\} \ . \tag{2.30}$$

We will assume that g_* is constant, which is a good approximation for temperature well above the QCD phase transition. In this case entropy conservation tells us that the temperature scales like a^{-1}, i.e. aT =const. We can factor out the expansion and define

$$Y \equiv \frac{n_X}{T^3} \qquad \text{and} \qquad Y_{(0)} \equiv \frac{n_X^{(0)}}{T^3} \tag{2.31}$$

to rewrite the differential equation for the number density in the convenient form

$$\frac{dY}{dt} = T^3 \langle \sigma v \rangle \left(Y_{(0)}^2 - Y^2 \right) \ . \tag{2.32}$$

The freeze-out process is characterised by the mass m_X of the particle X. Thus it is convenient to express the temperature in terms of m_X as follows

$$x = \frac{m_X}{T} \ . \tag{2.33}$$

In the radiation dominated era, the first Friedmann equation can be written as

$$H(T) = \left(\frac{1}{2t} \right) = \sqrt{\frac{8\pi G}{3} \frac{\pi^2}{30} g_*^\rho(T)} T^2 = \sqrt{\frac{8\pi^3 G}{90} g_*^\rho(T)} \frac{m_X^2}{x^2} = \frac{H(m_X)}{x^2} \tag{2.34}$$

using the effective relativistic degrees of freedom g_*^ρ and consequently the evolution equation can be rewritten as

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left(Y^2 - Y_{(0)}^2 \right) \tag{2.35}$$

with the generally quite large dimensionless parameter

$$\lambda \equiv \frac{m_X^3 \langle \sigma v \rangle}{H(m_X)} \ . \tag{2.36}$$

The cross section might depend on temperature, but in many theories it is constant or its temperature dependence can be neglected. In the following we will assume it to be constant.

There is no general analytic solution. However we can obtain an approximate analytic solution. As the constant λ is generally large, the abundance Y of the particle X will track its equilibrium value $Y_{(0)}$. However at late times for $T \ll m_X$, i.e. $x \gg 1$, when the equilibrium abundance is exponentially suppressed, we can neglect $Y_{(0)} \ll Y$ and find at late times

$$\frac{dY}{dx} \simeq -\frac{\lambda}{x^2} Y^2 \ , \tag{2.37}$$

which can be solved analytically. Integrating the solution from freeze-out, x_f to late times $x = \infty$, we obtain

$$Y_\infty = Y(x = \infty) \simeq \frac{x_f}{\lambda} \ . \tag{2.38}$$

An analytic estimate for the freeze-out temperature $T_f = m_X / x_f$ can be obtained from considering the size of the coefficient

$$\left. \frac{\lambda Y_{(0)}}{x} \right|_{\text{fo}} \simeq 1 \tag{2.39}$$

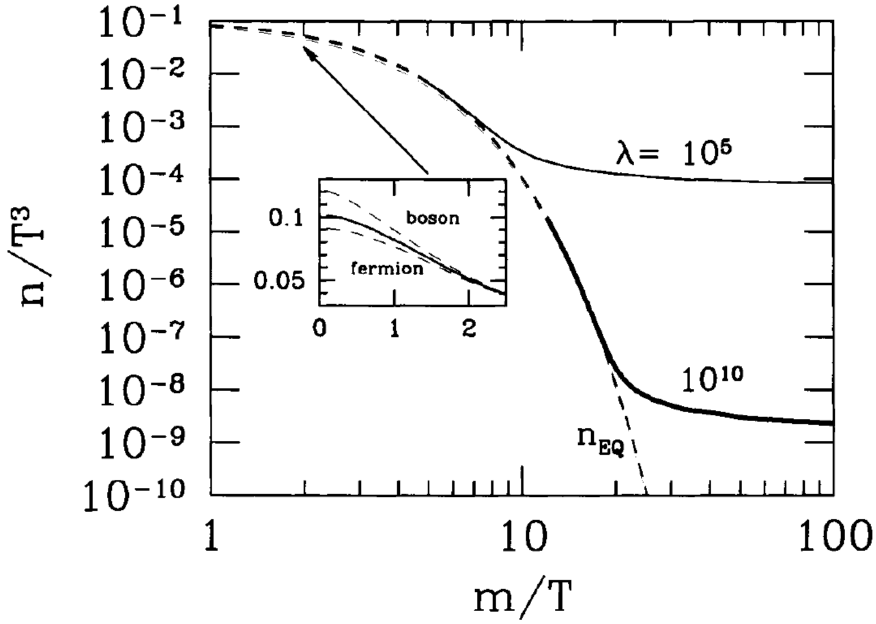

Figure 6: Dark matter freeze-out for $\lambda = 10^{10}$. Taken from Dodelson[1].

in the rescaled evolution equation

$$\frac{x}{Y_{(0)}}\frac{dY}{dx} = -\frac{\lambda Y_{(0)}}{x}\left(\left(\frac{Y}{Y_{(0)}}\right)^2 - 1\right) . \tag{2.40}$$

This results in an implicit equation for the freeze-out temperature

$$H(m_X) = x_f^2 \langle\sigma v\rangle\, n^{(0)}(x_f) \simeq \frac{g_X m_X \langle\sigma v\rangle}{(2\pi)^{3/2}} x_f^{1/2} e^{-x_f} . \tag{2.41}$$

Typical values for x_f are a few times 10. See Fig. 6.

Finally we want to obtain the energy density in the particle X. At temperature T_1 after the abundance Y reached its asymptotic value Y_∞, the number density is given by $Y_\infty T_1^3$. For later times, the number density scales like a^{-3}. Using entropy conservation

$$g_*^s(T_0)(a_0T_0)^3 = g_*^s(T_1)(a_1T_1)^3 \tag{2.42}$$

similar to the case of CMB photons and neutrinos we can relate the temperatures and find for the energy density today

$$\rho_X = m_X Y_\infty T_0^3 \left(\frac{a_1T_1}{a_0T_0}\right)^3 = m_X Y_\infty T_0^3 \frac{g_*^s(T_0)}{g_*^s(T_1)} \tag{2.43}$$

with $g_*^s(T_0)/g_*^s(T_1) \sim 1/30$. Finally we can express the energy density in terms of the critical energy density and find

$$\Omega_X h^2 = \frac{x_f}{\lambda}\frac{m_X T_0^3 h^2}{\rho_{cr}}\frac{g_*^s(T_0)}{g_*^s(T_1)} = 0.3\frac{x_f}{10}\sqrt{\frac{g_*^\rho(m_X)}{100}}\frac{10^{-39}\text{cm}^2}{\langle\sigma v\rangle} . \tag{2.44}$$

This is a remarkable result, which nicely ties in with particle physics, because the cross section needed

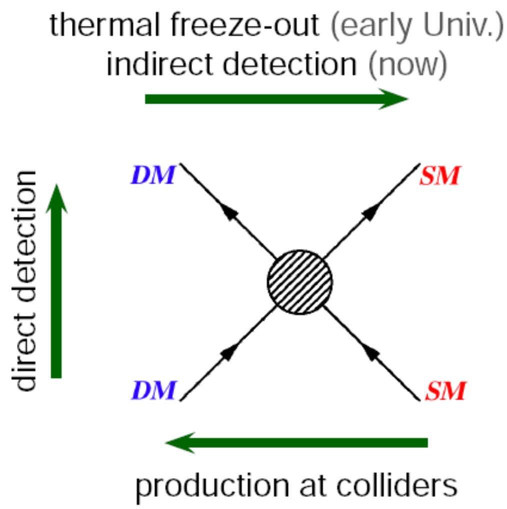

Figure 7: Crossing symmetry. Taken from `http://www.mpi-hd.mpg.de/lin`.

to obtain the correct relic abundance for a particle X with masses of ~ 100 GeV is of order of the weak-interaction cross section G_F^2. This coincidence is often called *WIMP miracle*, because a weakly interacting massive particle (WIMP) automatically obtains the correct abundance via freeze-out to explain dark matter. They naturally appear in many theories beyond the Standard Model (SM) of particle physics, like the lightest supersymmetric particle (LSP) in the minimal supersymmetric SM. There is a big experimental effort to search for these particles using all possible means: colliders, direct and indirect detection experiments. All three possible channels are related via crossing symmetry with the cross section relevant for dark matter pair annihilation in the early Universe, as it is shown in Fig. 7. WIMPs are particularly constrained by direct detection searches as shown in Fig. 8.

2.4 Big Bang Nucleosynthesis (BBN)

There are many processes contributing to BBN and a detailed discussion is beyond the scope of this lecture. We will only focus on the production of deuterium and give some general comments in the accompanying slides.

The condition for nuclear statistical equilibrium (2.28) for the process $n + p \leftrightarrow D + \gamma$ can be rewritten as follows

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} , \tag{2.45}$$

where we assumed $n_\gamma = n_\gamma^{(0)}$. Using the equilibrium number density given in Eq. (2.24), we can rewrite

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{2\pi m_D}{m_n m_p T} \right)^{3/2} e^{(m_n + m_p - m_D)/T} \tag{2.46}$$

with the masses $m_{p,n,D}$ for the proton, neutron and deuterium respectively. The factor of 3/4 originates from the internal degrees of freedom. Note that the deuterium is in a triplet state of spin. The combination of masses in the exponent is exactly the binding energy of deuterium B_D. In the prefactor

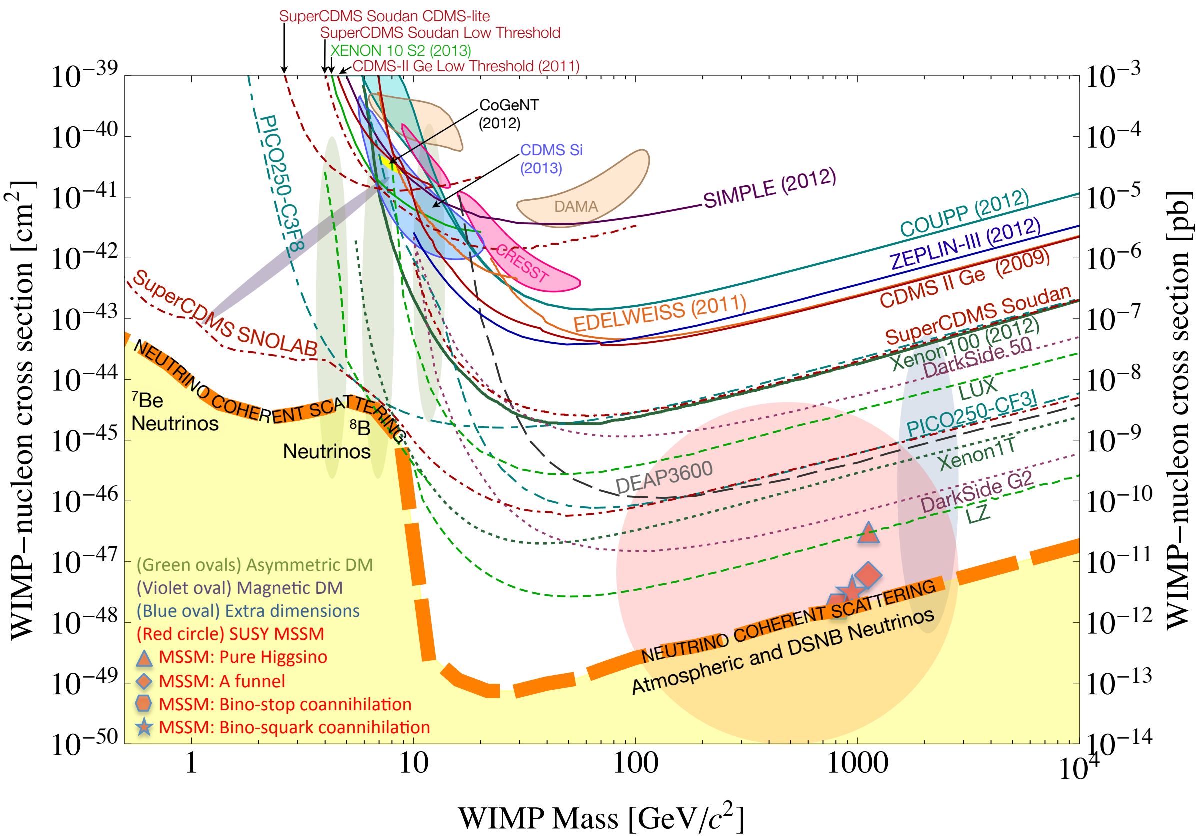

Figure 8: Dark matter direct detection.[5]

we can use $m_p \approx m_n \approx m_D/2$. Thus we obtain

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{B_D/T} \tag{2.47}$$

We can finally relate our result to the baryon density. The number densities of both neutron and proton are proportional to the baryon density n_b. Using $n_b \approx \eta_b n_\gamma = 2\eta_b T^3/\pi^2$ and dropping the numerical factors we obtain

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T} \ . \tag{2.48}$$

Note that $\eta_b \sim 10^{-10}$, which suppresses the production of deuterium (and also heavier elements) substantially below the binding energy of the nucleus. The prefactor dominates as long as the temperature is not much smaller than the binding energy $B_D = 2.22$ MeV.

3 The Boltzmann Equation

In the last section we studied the integrated form of the Boltzmann equation. However in order to study anisotropy we have to take the momentum dependence into account and study the unintegrated form of the Boltzmann equation,

$$\frac{df}{d\lambda} = C'[f] \tag{3.1}$$

with the distribution function $f = f(\vec{x}, \vec{p}, t)$. The left-hand side gives the change of the distribution function with respect to the affine parameter λ, which we introduced previously and $C'[f]$ is the collision term taking into account any interactions.

We will again use the momentum four-vector to define the affine parameter λ following Eq. (1.20). Thus we obtain

$$\frac{df}{dt} = \frac{1}{E}C'[f] = C[f] \tag{3.2}$$

which is exactly Eq. (4.1) in Dodelson[1].

The Boltzmann equations generally connect the different components of the Universe. Electrons and protons are coupled via the Coulomb interaction, photons and electrons[3] via Compton scattering. All particle species are coupled to the metric. See Fig. 9. In the following we will assume that the perturbations are small and expand all quantities to first order in the small perturbations.

3.1 Metric

As we want to study inhomogeneities and anisotropies, we also have to take a perturbation to the metric into account. We will consider perturbations to the flat FRW metric and restrict ourselves to scalar perturbations and do not consider vector or tensor perturbations. The metric in conformal Newtonian gauge is given by

$$ds^2 = -(1 + 2\Psi(\vec{x}, t))dt^2 + a^2(t)(1 + 2\Phi(\vec{x}, t))\delta_{ij}dx^i dx^j \ . \tag{3.3}$$

Ψ is the Newtonian potential and Φ, the perturbation of the spatial curvature. See exercise 3 in chapter 2 of Dodelson[1] to understand better the physical meaning of the two perturbations.

[3] Compton scattering between protons and photons is suppressed by the larger mass of a proton compared to an electron.

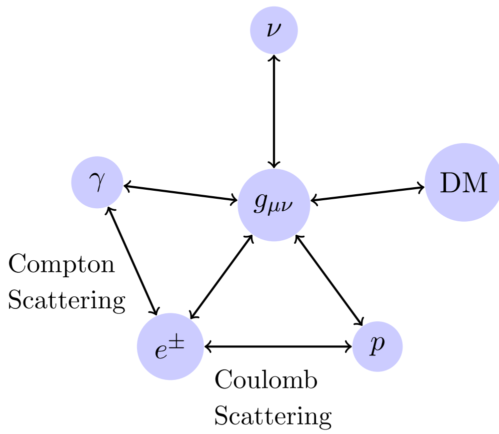

Figure 9: The network of Boltzmann and Einstein equations

3.2 Collisionless Boltzmann Equation for Photons

Not all components of the four-momentum

$$P^{\mu} \equiv \frac{dx^{\mu}}{d\lambda} \tag{3.4}$$

are independent. As photons are massless particles, we can use the dispersion relation

$$0 = g_{\mu\nu} P^{\mu} P^{\nu} = -(1 + 2\Psi) \left(P^0\right)^2 + p^2 \tag{3.5}$$

with

$$p^2 = g_{ij} P^i P^j = a^2 (1 + 2\Phi) \delta_{ij} P^i P^j \tag{3.6}$$

to express the time-component of P^{μ} in terms of the spatial components

$$P^0 = \frac{p}{\sqrt{1 + 2\Psi}} \doteq p(1 - \Psi) \tag{3.7}$$

to first order in Ψ. Note that an overdense region has $\Psi < 0$ and thus a photon moving out of the potential well will lose energy, i.e. redshift. The independent parameters are thus the position x^i, the momentum p and the direction of the momentum $\hat{p}^i$ satisfying $\hat{p}^i \hat{p}^j \delta_{ij} = 1$. The comoving momentum $P^i = C\hat{p}^i$ are proportional to the momentum direction $\hat{p}^i$. Using Eq. (3.6) we find

$$P^i = p\hat{p}^i \frac{1 - \Phi}{a} \,. \tag{3.8}$$

We write the total derivative as the sum of the partial derivatives

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} \,. \tag{3.9}$$

Note that we neglected the partial derivative with respect to $\hat{p}^i$, since it is second order in the small perturbation. Now we have to reexpress the different terms. Using the chain rule we find for

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{P^i}{P^0} \doteq \frac{\hat{p}^i}{a} (1 - \Phi + \Psi) \tag{3.10}$$

to first order. The time derivative of the momentum can be obtained from the zeroth component of the geodesic equation

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta \tag{3.11}$$

The left-hand side can be further evaluated to

$$\frac{dP^0}{d\lambda} = p(1-\Psi)\frac{d}{dt}\left(p(1-\Psi)\right) = p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi) - p\frac{d\Psi}{dt}\right] \tag{3.12}$$

$$= p(1-\Psi)\left[\frac{dp}{dt}(1-\Psi) - p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)\right] \doteq p(1-2\Psi)\left[\frac{dp}{dt} - p\left(\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right)\right] , \tag{3.13}$$

where we used Eq. (3.10). In order to evaluate the right-hand side, we have to evaluate the Christoffel symbol first

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left[\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\frac{P^\alpha P^\beta}{p} = \frac{g^{00}}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p} \tag{3.14}$$

$$= \frac{(-1+2\Psi)}{2}\left\{\frac{\partial g_{00}}{\partial t}\frac{P^0 P^0}{p} + \left[2\frac{\partial g_{0i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial t}\right]\frac{P^i P^j}{p} + 2\frac{\partial g_{00}}{\partial x^i}\frac{P^0 P^i}{p}\right\} \tag{3.15}$$

$$\doteq p\,(1-2\Psi)\left(H + \frac{\partial\Psi}{\partial t} + 2\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Phi}{\partial t}\right) \tag{3.16}$$

Combining everything we obtain

$$\frac{1}{p}\frac{dp}{dt} - \frac{\partial\Psi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = -H - \frac{\partial\Psi}{\partial t} - 2\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} - \frac{\partial\Phi}{\partial t} \tag{3.17}$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} . \tag{3.18}$$

This equation describes the change in photon momentum. An overdense region has $\Phi > 0$ and $\Psi < 0$. The first term accounts for the redshift due to the expansion, the second term states that a photon loses energy in a deepening potential well and a photon moving into a potential well gains energy.

Finally we have to consider the distribution function. Photons are bosons and thus are described by a Bose-Einstein distribution in equilibrium. We parameterize the perturbation as a change of the variation of the temperature $\Theta(\vec{x}, \hat{p}, t) = \delta T/T$, i.e.

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left(\frac{p}{T(t)(1+\Theta(\vec{x},\hat{p},t))}\right) - 1\right]^{-1} . \tag{3.19}$$

Note that we assume that Θ does not depend on the magnitude of the photon momentum. This is a good approximation for small-angle Compton scattering. Thus to leading order we find

$$f = f^{(0)} + \frac{\partial f^{(0)}}{\partial T}T\Theta = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \tag{3.20}$$

using that we can change the differentiation of the exponent from T to p.

After expanding each term in the small perturbations, we can now collect our results in Eqs.(3.9), (3.10), (3.18), and (3.20) and are now able to study the collisionless Boltzmann equation. At zeroth order we obtain

$$0 = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} - Hp\frac{\partial f^{(0)}}{\partial p} = -\left(\frac{dT/dt}{T} + \frac{da/dt}{a}\right)p\frac{\partial f^{(0)}}{\partial p} \tag{3.21}$$

and consequently we recover the well-known result $aT = \text{const}$.

At first order we evaluate each of the terms on the right-hand side of Eq. (3.9) separately. The first term evaluates to

$$-p\frac{\partial}{\partial t}\left(\frac{\partial f^{(0)}}{\partial p}\Theta\right) = -p\frac{\partial\Theta}{\partial t}\frac{\partial f^{(0)}}{\partial p} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p} = -p\frac{\partial\Theta}{\partial t}\frac{\partial f^{(0)}}{\partial p} + \frac{dT/dt}{T}p\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) , \quad (3.22)$$

the second term only consists of one term

$$-p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial x^i}\frac{\hat{p}^i}{a} \quad (3.23)$$

and the third term

$$Hp\Theta\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}\right) - \frac{\partial f^{(0)}}{\partial p}p\left(\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right) \quad (3.24)$$

partly cancels the first term using the zeroth order equation. Thus we obtain

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] . \quad (3.25)$$

The last two account for the effect of gravity, while the first two are entirely due to the change in the distribution function. It is useful to go to Fourier space for the spatial coordinates and to use conformal time η. Defining

$$\Theta(\vec{x}) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k}) \quad (3.26)$$

and using a dot to indicate a derivative with respect to conformal time ($\dot{\Theta} = d\Theta/d\eta$) we obtain

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -\frac{p}{a}\frac{\partial f^{(0)}}{\partial p}\left[\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi}\right] \quad (3.27)$$

with $k\mu \equiv \hat{p}\cdot\vec{k}$.

3.3 Collision Terms: Compton Scattering

Photons are interacting with electrons via Compton scattering

$$e^-(q) + \gamma(p) \leftrightarrow e^-(q') + \gamma(p') . \quad (3.28)$$

The collision term for Compton scattering is given by

$$C[f(\vec{p})] = \frac{1}{p}\int d\Pi_q d\Pi_{q'} d\Pi_{p'}|\mathcal{M}|^2\,(2\pi)^4\,\delta^{(4)}(p+q-p'-q')\left(f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\right) . \quad (3.29)$$

The integral over d^3q' is trivially evaluated using the delta function over the three-momenta. We want to evaluate the collision term in the limit of non-relativistic electrons, i.e. $E(\vec{q}) = m_e + q^2/2m_e$ and consequently also photons with small momenta $|p| \sim T \ll m_e$. Thus the energy transfer is

$$E_e(\vec{q}) - E_e(\vec{q}+\vec{p}-\vec{p}') \simeq \frac{(\vec{p}'-\vec{p})\cdot\vec{q}}{m_e} , \quad (3.30)$$

where we used that the momentum of the photons is much smaller than the one of the electrons, $|p|, |p'| \ll |q|$. As the electrons are non-relativistic, their velocity $q/m_e \sim v_b \ll 1$ is small and similar to the baryon velocity v_b. Thus the energy transfer is of order of $Tq/m_e \sim Tv_b$. We can thus expand the energy delta function around zero energy transfer

$$\delta(p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q}+\vec{p}-\vec{p}')^2}{2m_e}) \simeq \delta(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'} \,. \tag{3.31}$$

Similarly we can make the approximation that the momentum distribution of the electrons does not change, i.e. $f_e(\vec{q}+\vec{p}-\vec{p}') \simeq f_e(\vec{q})$. Thus the collision term reads

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p}\int \frac{d^3q}{(2\pi)^3} f_e(\vec{q}) \int \frac{d^3p'}{(2\pi)^3 p'} |\mathcal{M}|^2 \left(\delta(p-p') + \frac{(\vec{p}-\vec{p}')\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right)\left(f(\vec{p}') - f(\vec{p})\right) \tag{3.32}$$

Following Dodelson, we will neglect the polarisation of the photons and average over it for simplicity of the discussion. A proper treatment should include the polarisation. The matrix element for Compton scattering after summing over the polarisation of the incoming and outgoing photons is given by

$$|\mathcal{M}|^2 = 4\pi\sigma_T m_e^2\left(2 + P_2(\hat{p}\cdot\hat{p}')\right) = 4\pi\sigma_T m_e^2\left(2 + \frac{4\pi}{5}\sum_{m=-2}^{2} Y_{2m}(\hat{p})Y_{2m}^*(\hat{p}')\right) \tag{3.33}$$

with the second Legendre polynomial $2P_2(x) = 3x^2 - 1$ and the corresponding spherical harmonics Y_{lm}. Note that the integral over the azimuthal angle φ vanishes for $Y_{lm}(\hat{p}') \propto e^{im\varphi}$ for $m \neq 0$. Thus we find using $Y_{20}(\hat{p}) = \sqrt{5/4\pi}P_2(\hat{p}\cdot\hat{k})$ for some fixed $\hat{k}$

$$|\mathcal{M}|^2 = 4\pi\sigma_T m_e^2\left(2 + P_2(\mu)P_2(\mu')\right) \tag{3.34}$$

with $\mu' = \hat{p}'\cdot\hat{k}$ The q-integration yields the number density n_e and the velocity $n_e\vec{v}_b$ in case of the factor $\vec{q}$. We expand the distribution functions (3.20)

$$f(p') - f(p) = f^{(0)}(p') - f^{(0)}(p) - p'\frac{\partial f^{(0)}(p')}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}(p)}{\partial p}\Theta(\hat{p}) \,. \tag{3.35}$$

It is easy to see that the zeroth order term vanishes, because the product of the delta-function with the difference of the equilibrium distributions vanishes, when integrated over p'. The leading non-vanishing term is given by the sum of the terms proportional to v_b and Θ, which we evaluate separately. Using spherical coordinates for the p' integral we obtain

$$C_\Theta[f(\vec{p})] = \frac{\sigma_T n_e}{8\pi p}\int d\Omega'\left(2 + P_2(\mu)P_2(\mu')\right)\int p'dp'\delta(p-p')\left(-p'\frac{\partial f^{(0)}(p')}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}(p)}{\partial p}\Theta(\hat{p})\right) \tag{3.36}$$

$$. \tag{3.37}$$

The momentum integral of C_Θ can be directly evaluated with the delta function and we obtain

$$C_\Theta[f(\vec{p})] = \sigma_T n_e p\frac{\partial f^{(0)}(p)}{\partial p}\left(\Theta(\hat{p}) - \Theta_0 + \frac{1}{2}P_2(\mu)\Theta_2\right) \,, \tag{3.38}$$

where we defined the moments

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu'}{2} P_l(\mu')\Theta(\mu') \; . \tag{3.39}$$

The second term C_{v_b}

$$C_{v_b}[f(\vec{p})] = \frac{\sigma_T n_e}{8\pi p} \int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right) \int p' dp' (\vec{p} - \vec{p}') \cdot \vec{v}_b \frac{\partial \delta(p-p')}{\partial p'} \left(f^{(0)}(p') - f^{(0)}(p)\right) \tag{3.40}$$

can be similarly evaluated by first noticing that the angular integral over $P_2(\mu')\vec{p}' \cdot \vec{v}_b$ is odd and vanishes and by using partial integration to remove the derivative from the delta function

$$\begin{aligned} C_{v_b}[f(\vec{p})] &= -\frac{\sigma_T n_e}{8\pi p} \int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right) \int dp' \delta(p-p') \frac{\partial}{\partial p'} \left[p' \vec{p} \cdot \vec{v}_b \left(f^{(0)}(p') - f^{(0)}(p)\right)\right] && (3.41) \\ &= -\frac{\sigma_T n_e}{8\pi p} \int d\Omega' \left(2 + P_2(\mu)P_2(\mu')\right) \int dp' \delta(p-p') \left[p' \vec{p} \cdot \vec{v}_b \frac{\partial f^{(0)}(p')}{\partial p'}\right] && (3.42) \\ &= -\sigma_T n_e p \frac{\partial f^{(0)}(p)}{\partial p} \hat{p} \cdot \vec{v}_b \; . && (3.43) \end{aligned}$$

Collecting both term, we obtain for the collision term

$$C[f(\vec{p})] = \sigma_T n_e p \frac{\partial f^{(0)}(p)}{\partial p} \left[\Theta(\hat{p}) - \Theta_0 + \frac{1}{2} P_2(\mu)\Theta_2 - \hat{p} \cdot \vec{v}_b\right] \; . \tag{3.44}$$

Finally we take the Fourier transform of the collision term. Typically we will assume that the velocity points in the same direction as $\vec{k}$, i.e. so the Fourier transform of $\hat{p} \cdot \vec{v}_b$ becomes $\tilde{v}_b \mu$. Hence the Fourier transform (of the spatial directions) is given by

$$C[f(\vec{p})] = \sigma_T n_e p \frac{\partial f^{(0)}(p)}{\partial p} \left[\tilde{\Theta}(\hat{p}) - \tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 - \tilde{v}_b \mu\right] \tag{3.45}$$

3.4 Boltzmann Equation for Photons

Combining the result for the collisionless Boltzmann equation (3.27) with the collision term (3.45), we obtain the Boltzmann equation for photons coupled to non-relativistic electrons

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = \sigma_T n_e a \left[\tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 + \tilde{v}_b \mu - \tilde{\Theta}\right] \; . \tag{3.46}$$

Note that the different Fourier modes do not mix and thus evolve independently. This only holds in the linear regime. If the perturbations can be large, as it is the case for matter, the linear approximation breaks down and different Fourier modes will couple. Finally we use the optical depth

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a \; , \tag{3.47}$$

which characterises how much light is absorbed by the electrons, i.e. the intensity $I(\eta_0)$ today compared to the intensity at η ,

$$I(\eta_0) = I(\eta) e^{-\tau} \tag{3.48}$$

to write the Boltzmann equation describing photons

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau} \left[\tilde{\Theta}_0 + \frac{1}{2} P_2(\mu)\tilde{\Theta}_2 + \tilde{v}_b \mu - \tilde{\Theta}\right] \; . \tag{3.49}$$

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